Chapter 9 - Day 2

Recall from Chapter 3

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \lim_{n\to\infty} \frac{\text{highest order term of } p(n)}{\text{highest order term of } q(n)}$$

<u>Ex</u>: $\displaystyle \lim_{n\to\infty} \frac{10n^3+2n+1}{5n^3+3n^2}$

$$= \lim_{n\to\infty} \frac{10n^3}{5n^3}$$

$$= \lim_{n\to\infty} \frac{10}{5}$$

$$= \frac{10}{5} = \boxed{2}$$

Ex: $\lim_{n\to\infty} \frac{(3n+2)^2}{7n^2+6n+1}$

$$= \lim_{n\to\infty} \frac{9n^2}{7n^2} = \lim_{n\to\infty} \frac{9}{7} = \boxed{\frac{9}{7}}$$

Ex: $\lim_{n\to\infty} \frac{n^4+3n^2+1}{n^3+4n}$

$$= \lim_{n\to\infty} \frac{n^4}{n^3} = \lim_{n\to\infty} \frac{n}{1}$$

"unbounded"

$$= \boxed{\text{DNE}}$$

Our definition of the definite integral using right endpoints and subintervals of equal length,

then $\Delta x = \dfrac{b-a}{n}$

and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(a+k\Delta x)\cdot\Delta x$$

Ex: Evaluate the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k+9}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{n} \sum_{k=1}^{n} (k+9)$$ factor out $\frac{1}{n}$

$$= \lim_{n\to\infty} \frac{1}{n^2} \left(\sum_{k=1}^{n} k + \sum_{k=1}^{n} 9 \right)$$ split sum

$$= \lim_{n\to\infty} \frac{1}{n^2} \left(\frac{n(n+1)}{2} + 9(n) \right)$$ summation formulas

$$= \lim_{n\to\infty} \left(\frac{n(n+1)}{2n^2} + \frac{9n}{n^2} \right)$$

$$= \lim_{n\to\infty} \frac{n(n+1)}{2n^2} + \lim_{n\to\infty} \frac{9n}{n^2}$$

$$= \lim_{n\to\infty} \frac{n+1}{2n} + \lim_{n\to\infty} \frac{9}{n}$$

$$= \lim_{n\to\infty} \frac{n}{2n} + \lim_{n\to\infty} \frac{9}{n}$$

$$= \frac{1}{2} + 0 = \boxed{\frac{1}{2}}$$

Ex: The integral $\int_0^5 x^2\,dx$

is computed as the limit of the sum $\sum_{k=1}^{n} \frac{A}{n}\left(k\,\frac{A}{n}\right)^2$

What value of A must appear in the sum?

$$\int_0^5 x^2\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_n)\cdot\Delta x$$

when $\Delta x = \frac{5-0}{n} = \frac{5}{n}$

and $x_n = 0 + k\Delta x = k\left(\frac{5}{n}\right)$

thus

$$\int_0^5 x^2\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(k\cdot\frac{5}{n}\right)^2\cdot\frac{5}{n}$$

now match terms! $\boxed{A=5}$

Ex: $\int_1^2 f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{3}{n}\left(\frac{n+k}{n}\right)^2$

What is $f(x)$?

$\int_1^2 f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_n)\cdot \Delta x$

where $\Delta x = \frac{2-1}{n} = \frac{1}{n}$

and $x_n = 1 + k\left(\frac{1}{n}\right)$

then $\int_1^2 f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(1+\frac{k}{n}\right)\cdot \frac{1}{n}$

now play the matching game!

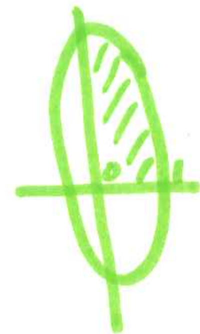
$\frac{1}{n} f\left(1+\frac{k}{n}\right) = \frac{3}{n}\left(\frac{n+k}{n}\right)^2$

$f\left(1+\frac{k}{n}\right) = 3\left(\frac{n+k}{n}\right)^2$

$f\left(\frac{n+k}{n}\right) = 3\left(\frac{n+k}{n}\right)^2$

thus $\boxed{f(x) = 3x^2}$

<u>Ex</u>: The area of the ellipse $49x^2+y^2=49$ is 7π. Evaluate the integral $\int_0^1 \sqrt{49-49x^2}\,dx$



$$49x^2+y^2=49$$
$$y^2=49-49x^2$$
$$y=\sqrt{49-49x^2}$$ is the top half of the ellipse

Integrating from 0 to 1 finds the area under the right half of the top portion.

$$\int_0^1 \sqrt{49-49x^2} = \frac{1}{4}(\text{Area of Ellipse})$$
$$= \frac{1}{4}(7\pi) = \boxed{\frac{7\pi}{4}}$$